# Estimating Filtering Errors Using the Peano Kernel Theorem

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*Abstract: The Peano Kernel Theorem is introduced and a frequency domain derivation is given. It is demonstrated that the application of this theorem yields simple and accurate formulas for estimating the error introduced into a signal by filtering it to reduce noise.*

## Introduction

This paper deals with the following problem: A signal $y$(t) is measured with

$$y(t) = x(t) + n(t), \tag{1}$$

where $x(t)$ is the true signal and $n(t)$ is noise. Actual measurements are made with discrete time, but continuous time will be used here, because the mathematics is more straightforward. It is assumed that the sampling density is high enough that the difference between the integrals given here and the corresponding sums is sufficiently accurate. The signal, $x$, is estimated by filtering $y$ to reduce the effects of noise. In other words

$$\hat{x}(t) = \tilde{g} * y(t) = \tilde{g} * x(t) + \tilde{g} * n(t)\,, \tag{2}$$

where the asterisk denotes convolution, $\hat{x}(t)$ is the estimate of $x(t)$, and $\tilde{g}(t)$ is the impulse response of the filter. The reason for the tilde over the function $g$(t) will be explained later. Although this example uses convolution, the analysis in the paper applies equally well to time-varying filters. The error is the difference between the true signal and its estimate. It can be decomposed as follows:

$$\hat{x}(t) - x(t) = \tilde{g} * x(t) - x(t) + \tilde{g} * n(t) = (\tilde{g} - \delta) * x(t) + \tilde{g} * n(t) = e_F(t) + e_R(t)\,, \tag{3}$$

where $\delta$ is the Dirac delta function. The second term above, $e_R$, is the random error due to the noise, and its magnitude can be easily estimated from knowledge of the noise spectrum and the impulse response of the filter [1,2]. This paper is concerned with estimating $e_F(t)$, the filtering error. The estimation is via the Peano Kernel Theorem (PKT) [3,4]. Although the PKT has been used in numerical analysis for a long time, it has not been used for estimating the errors due to filtering in measurement applications. To apply the PKT, the problem must be reformulated in terms of functionals. This is done by concentrating on a particular value, $t = t_0$, and estimating $e_F(t_0)$. Without any loss of generality, we can take $t_0 = 0$, because any other value can be obtained by translation of the data. The reformulated problem is to estimate the error in approximating

$$F(x) = \int f(t)x(t)dt \text{ with } G(x) = \int g(t)x(t)dt. \tag{4}$$

In the above example $f$ is the delta function, and $g$ is the time reversal of the impulse response of the filter, i.e. $g(t) = \tilde{g}(-t)$.

## The Peano Kernel Theorem

The PKT deals with approximations of the type given in (4). Writing $E(x) = G(x) - F(x)$, it asserts that

$$\begin{gathered}\text{If } E(x) = 0 \text{ whenever } x \text{ is a polynomial of degree } n-1 \text{ or less,}\\ \text{then there is a function } k_n(t) \text{ such that } E(x) = \int k_n(t)x^{(n)}(t)dt.\end{gathered} \tag{5}$$

The function, $x^{(n)}(t)$, is the $n^{\text{th}}$ derivative of $x(t)$. Formulas for calculating $k_n(t)$ are given in [2] and [5], but a different, frequency domain, approach will be used here.

The condition of the theorem, that $E(x) = 0$ for $x$ a polynomial of degree $n$-1 or less, is equivalent to

$$\int g(t)dt = 1, \text{ and } \int t^k g(t) = 0 \text{ for } 1 \le k \le n-1. \tag{6}$$

Unless otherwise stated, we are assuming that $f(t) = \delta(t)$. Condition (6) can be stated in the frequency domain as

$$\hat{g}(0) = 1, \text{ and } \hat{g}^{(k)}(0) = 0 \text{ for } 1 \le k \le n-1. \tag{7}$$

The formula for $E(x)$ can be written in the frequency domain using Parseval's relation:

$$E(x)=\int (g(t)-\delta(t))x(t)dt=\frac{1}{2\pi}\int \overline{(\hat{g}(\omega)-1)}\hat{x}(\omega)d\omega$$
$$=\frac{1}{2\pi}\int \frac{\overline{(\hat{g}(\omega)-1)}}{\overline{(-j\omega)^n}}(j\omega)^n\hat{x}(\omega)d\omega=\frac{1}{2\pi}\int \overline{\hat{k}_n(\omega)}\widehat{x^{(n)}}(\omega)d\omega. \tag{8}$$

The hat over a function indicates the Fourier transform of the function, and the bar over an expression indicates the complex conjugate of the expression. From this it can be seen that

$$\hat{k}_n(\omega)=\frac{\hat{g}(\omega)-1}{(-j\omega)^n}. \tag{9}$$

The conditions in (7) guarantee that the function specified in (9) remains bounded at $\omega=0$. The above essentially constitutes a frequency-domain proof of the PKT. In this paper a filter is said to have $n^{th}$ order accuracy if (7) is satisfied and is said to have maximum order of accuracy $n$ if $n$ is the largest integer for which (7) is satisfied.

## Examples of Peano kernels

For the first example we let

$$\hat{g}(\omega)=\frac{1}{1+(\omega T)^n}. \tag{10}$$

Here, $T$ is a scaling parameter that is proportional to the duration of the impulse response and inversely proportional to the bandwidth of the filter, and $n$ is an even integer. Using (9), the $n^{th}$ order kernel is given by

$$\hat{k}_n(\omega)=\frac{1}{(-j\omega)^n}\left[\frac{1}{1+(\omega T)^n}-\frac{1+(\omega T)^n}{1+(\omega T)^n}\right]=\frac{1}{(-j\omega)^n}\frac{-(\omega T)^n}{1+(\omega T)^n}$$
$$=j^nT^n\frac{1}{1+(\omega T)^n}=(-1)^{\frac{n}{2}}T^n\hat{g}(\omega). \tag{11}$$

In this case the Peano kernel is, except for its sign, the original filter multiplied by $T^n$.

For the second example let

$$g(t)=\begin{cases}\frac{1}{T} & \text{for } |t|\le\frac{T}{2} \\ 0 & \text{otherwise}\end{cases}. \tag{12}$$

This represents an average of the data over an interval of length $T$. The Fourier transform is given by

$$\hat{g}(\omega)=\frac{\sin\left(\omega T/2\right)}{\left(\omega T/2\right)}. \tag{13}$$

This has a maximum order of accuracy of 2, and from (9) the Fourier transform of the second order Peano kernel is

$$\hat{k}_2(\omega)=-\frac{\sin\left(\omega T/2\right)-\left(\omega T/2\right)}{\left(\omega T/2\right)\omega^2}. \tag{14}$$

Expanding the sin function in a power series to degree three and simplifying gives

$$\hat{k}_2(\omega)=\frac{T^2}{24}\hat{h}\left(\omega T/2\right),\text{ with }\hat{h}(0)=1 \tag{15}$$

The function $\hat{h}$ can be evaluated and plotted and seen to be a low pass filter.

## Estimating the filtering error

The analysis has shown that the error due to filtering with a filter, $g$, with maximum order $n$ is given by

$$e_F(t) = C_g T^n \breve{x}^{(n)}(t) \cong C_g T^n x^{(n)}(t), \tag{16}$$

where $C_g$ is an easily derived constant that depends on $g$, and $T$ is the scaling parameter of the filter. The symbol, $\breve{x}(t)$ , denotes the filtering of $x$ with a low pass filter that has a gain of 1 at low frequency—the normalized $n^{\text{th}}$ order Peano kernel. When comparing the error introduced by different filters one should compare filters that have the same effect in reducing the noise. For white noise this means comparing filters with the same equivalent noise bandwidth [2], given by

$$B_{eq} = \int_{-\infty}^{\infty} |g(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{g}(\omega)|^2 \, d\omega \ . \tag{17}$$

It is more convenient here to use the equivalent averaging time, given by

$$T_{eq} = 1 \big/ B_{eq} \, . \tag{18}$$

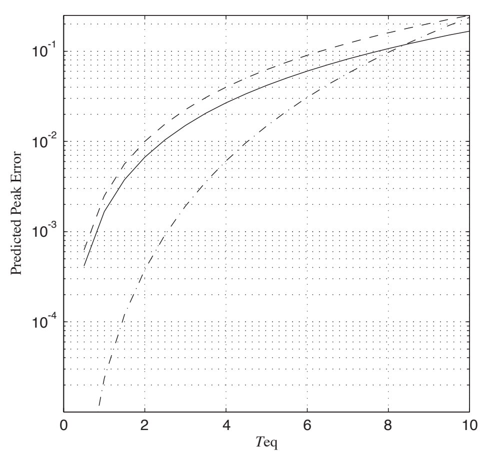
The error estimates for the previously given filters are given below:

| Filter | $T_{eq}$ | Error Formula | Error Formula |
|---|---|---|---|
| Averaging | $T$ | $\left(T^2 / 24\right) x^{(2)}(t)$ | $\left(T_{eq}^2 / 24\right) x^{(2)}(t)$ |
| $1/(1+(\omega T)^2)$ | $4T$ | $T^2 x^{(2)}(t)$ | $0.0625 T_{eq}^2 x^{(2)}(t)$ |
| $1/(1+(\omega T)^4)$ | $3.77T$ | $T^4 x^{(4)}(t)$ | $0.0049 T_{eq}^4 x^{(4)}(t)$ |

These filters were applied to the signal

$$x(t) = e^{-\frac{t^2}{50}} \, . \tag{19}$$

For this signal both the second and fourth derivatives are maximum at $t = 0$, the values being 0.04 and 0.0048, respectively. These values were used in the formulas above to calculate the predicted errors shown in Figure 1. The agreement between the actual error and the predicted error is very good.



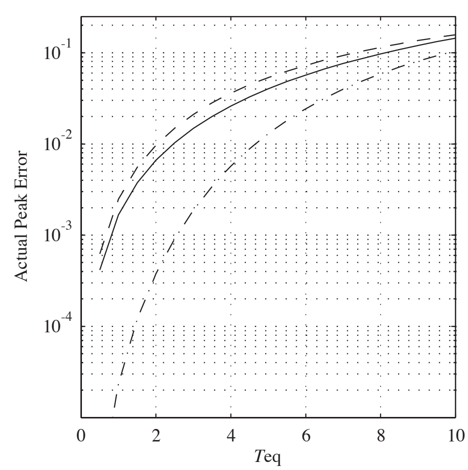




**Figure 1**. These plots show the predicted (from the formula in the table above) and the actual peak errors from applying the three filters to the signal given in (19). The solid line is the averaging filter, the dashed line the 2nd order rational filter. The lower curve is the fourth order filter. The actual errors are smaller than the predicted errors for larger values of $T_{eq}$, because the time window of the filter encloses smaller values of the derivatives.

## Conclusion

The Peano Kernel theorem was shown to provide simple and accurate estimates for the error caused by a filter. The estimates can be calculated using straightforward algebraic operations on the filter transfer function.

## References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